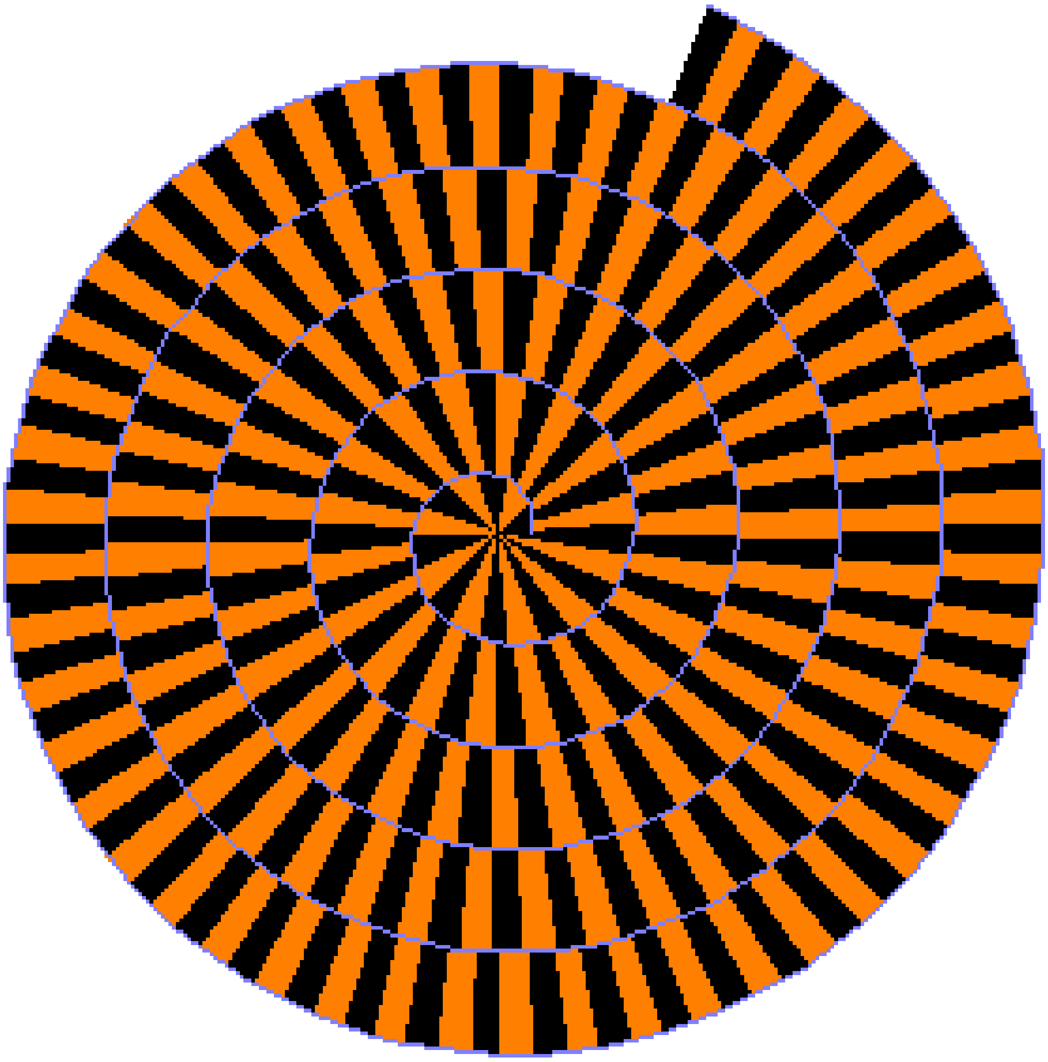
The Wheel of Theodrous



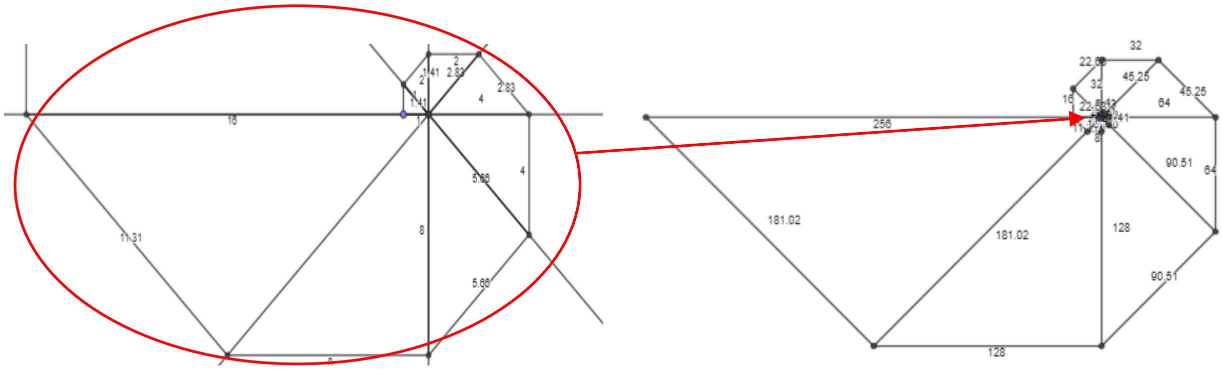
Introduction

In my grade 7 math class, I was fascinated by the nature of irrational numbers when I learnt that Hippasus, a Pythagorean philosopher, is supposed to have been drowned at sea for insisting the existence of irrational numbers, since Pythagoras and his followers simply did not accept irrational numbers for contradicting their strong belief, as they believed all numbers are able to be expressed as a fraction of two integers. Even nowadays, I am still intrigued by these special numbers, as whenever I fold square napkin diagonally, I wonder how to express a new base of the isosceles triangle. I realized that as long as the dimension of a napkin is not multiple of $\sqrt{2}$ units, which will usually be the case, the diagonal will always be irrational, since the ratio of the bases and hypotenuse of an isosceles triangle will always be n:n:n$\sqrt{2}$. Such approach to irrational numbers was a very exciting matter for me, since this meant that it is impossible to draw irrational length by itself, although there is no doubt that these numbers exist. Not being able to draw certain length of line without drawing other lines also indicates that the irrational numbers cannot be independently placed on a number line. The fact that we cannot draw, nor place irrational numbers on a number line, yet could clearly see them in real life – hence, there is no doubt that they exist - made me want to investigate more about it.

Model

To further explore this brainstorming idea, I created right angle triangles with base and side of 1 as an initial triangle, using GeoGebra. Next triangle used a hypotenuse of previous triangle as its base and the side and the diagram looked like an ammonite, or rough sketch of a spiral. Initially, the diagram stopped at the first revolution, but I went one more revolution and did not continue anymore, as it was apparent to see identical design of the wheel would repeat every revolution with increasing scale (see Figure 1). The two diagrams of figure 1 are in different scale but the size was adjusted to show that they follow identical shape.

Figure 1: Two diagrams of my initial model where the left diagram represents the first revolution, and the right diagram represents the first and second revolution



Then, I researched whether there was any relevant theory or similar-looking diagram in online, and found 'The Wheel of Theodorus' (hereafter, TWT). TWT can be basically drawn in almost identical way as my initial sketch, except the sides of triangles that are opposite to the angles from the origin are fixed to 1 unit. I was very amazed by the fact that TWT could reveal many mathematical concepts, including Archimedean spiral, π and, of course, irrational numbers (Fractal Kitty, 2021).

Constructing the wheel was an essential process, as the diagram would provide a visual representation for easier comprehension and would confirm any of the calculations related to its properties.

Note that the scale of the following diagrams is not identical, as the size of each diagram was adjusted for the sake of simplicity and saving pages.

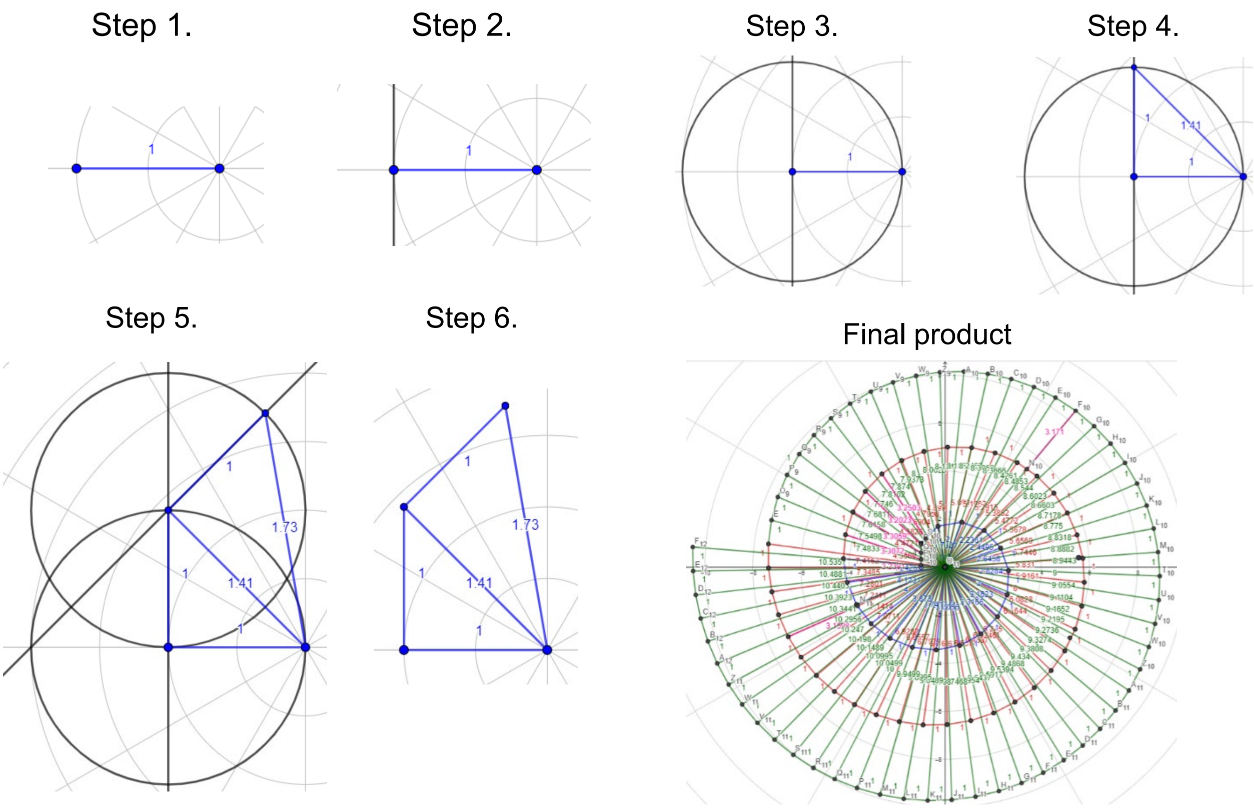
TWT was created by following steps in GeoGebra (see Figure 2):

1. Draw a segment with 1 unit from origin.
2. Draw a perpendicular line to the segment.

3. Draw a circle with a radius of 1 unit and centre being a vertex.
4. Draw a line from centre of the circle to the intersection of the perpendicular line and the circle. Then, connect the intersecting point to the origin.
5. Repeat step 2 to 4 with the hypotenuse being the new segment for the next triangle.
6. Hide all the perpendicular lines and circles.

Following the steps, TWT with 3 revolutions were produced (see Figure 2). This diagram was used to validate whether my findings were true.

Figure 2: 7 diagrams representing each step and the final product of TWT



Properties of TWT

Some basic properties were first explored as they were to be used to derive more complex properties.

Radius:

The radius, or rather the hypotenuse of a triangle, follows a clear pattern of $\sqrt{n+1}$, where n represents n^{th} triangle. This is because the first triangle has side lengths of 1 and 1, and from there, one of the side lengths is fixed to 1 unit.

$$Radius\ (hypotenuse\ of\ 1^{st}\ triangle) = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$Radius\ (hypotenuse\ of\ 2^{nd}\ triangle) = \sqrt{1^2 + \left(\sqrt{2}\right)^2} = \sqrt{3}$$

$$\vdots$$

$$Radius\ (hypotenuse\ of\ n^{th}\ triangle) = \sqrt{1^2 + \left(\sqrt{n}\right)^2} \quad \textbf{(1)}$$

$$= \sqrt{1+n}, for\ n \in \mathbb{Z}^+$$

Angle that indicates the progression of the wheel:

Firstly, the angle of any triangle in the wheel can be calculated as following, as the opposite length is fixed to 1 unit and hypotenuse side of n^{th} angle will always be the $\sqrt{n+1}$ (see equation 1).

$$\theta_n = \sin^{-1}\left(\frac{opposite\ of\ \ n^{th}\ triangle}{hypotenuse\ of\ n^{th}\ triangle}\right)$$

$$= \sin^{-1}\left(\frac{1}{\sqrt{n+1}}\right)$$

Hence, the sum of angles of all triangles up to n^{th} triangle will be:

$$\sum_{x=1}^{n} \sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right) \quad \textbf{(2)}$$

With the equation of the radius and the angle, the following table was created (all the calculated values were rounded to 5 significant figures for appropriate accuracy):

Table 1: Radii and angles of each triangle for the first two revolutions

n^{th} triangle	Radius	θ_n (rad)	n^{th} triangle	Radius	θ_n (rad)	n^{th} triangle	Radius	θ_n (rad)
1	1.4142	0.78540	19	4.4721	0.22551	37	6.1644	0.16294
2	1.7321	0.61548	20	4.5826	0.21999	38	6.2450	0.16082
3	2.0000	0.52360	21	4.6904	0.21485	39	6.3246	0.15878
4	2.2361	0.46365	22	4.7958	0.21006	40	6.4031	0.15682
5	2.4495	0.42053	23	4.8990	0.20557	41	6.4807	0.15492
6	2.6458	0.38760	24	5.0000	0.20136	42	6.5574	0.15310
7	2.8284	0.36137	25	5.0990	0.19740	43	6.63332	0.15133
8	3.0000	0.33984	26	5.1962	0.19366	44	6.7082	0.14963
9	3.1623	0.32175	27	5.2915	0.19013	45	6.7823	0.14798
10	3.3166	0.30628	28	5.3852	0.18678	46	6.8557	0.14639
11	3.4641	0.29284	29	5.4772	0.18360	47	6.9282	0.14484
12	3.6056	0.28103	30	5.5678	0.18059	48	7.0000	0.14335
13	3.7417	0.27055	31	5.6569	0.17771	49	7.0711	0.14190
14	3.8730	0.26116	32	5.7446	0.17497	50	7.1414	0.14049
15	4.0000	0.25268	33	5.8310	0.17235	51	7.2111	0.13912
16	4.1231	0.24498	34	5.9161	0.16985	52	7.2801	0.13780
17	4.2426	0.23794	35	6.0000	0.16745	53	7.3485	0.13651
18	4.3589	0.23148	36	6.0828	0.16515	54	7.4162	0.13525

From the table, I could see the trend where both the change in radius and the centre angle of a triangle were gradually decreasing as the wheel progresses. For example, the difference of radii between first and last values of the last column (θ_{37} and θ_{54}= 6.1644 and 7.4162) clearly decreased from that of the first column (θ_1 and θ_{18}=

1.4142 and 4.3589). Similarly, the centred angle of a triangle is constantly decreasing as can be seen from the first value and the last value of the table (0.78540 and 0.13525). Hence, limits at infinity were used to corroborate this inductive reasoning.

Limits at infinity for the change in radius:

As explained previously, the hypotenuse and base lengths of n^{th} triangle are $\sqrt{n+1}$ and $\sqrt{n}$ (see equation 1). Hence the change in radius will be the difference between those lengths, and this change will approach to be negligible as the wheel endlessly progresses.

$$\Delta r = \sqrt{n+1} - \sqrt{n}$$

$$\lim_{n\to\infty}\left(\sqrt{n+1} - \sqrt{n}\right) = \lim_{n\to\infty}\left(\sqrt{n+1} - \sqrt{n}\right) \times \frac{\left(\sqrt{n+1} + \sqrt{n}\right)}{\left(\sqrt{n+1} + \sqrt{n}\right)}$$

$$= \lim_{n\to\infty}\frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{1}{\sqrt{\infty+1} + \sqrt{\infty}}$$

$$= \frac{1}{\infty}$$

$$= 0$$

Limits at infinity for the angle:

Same process as the calculation above was done for the equation of angle of n^{th} triangle (see equation 2).

$$\theta_\infty = \lim_{n\to\infty}\left(\sin^{-1}\left(\frac{1}{\sqrt{n+1}}\right)\right)$$

$$= \sin^{-1}\left(\frac{1}{\sqrt{\infty+1}}\right)$$

$$= 0 + 2k\pi$$

$$= 0, as\ 0 \leq \theta < 2\pi$$

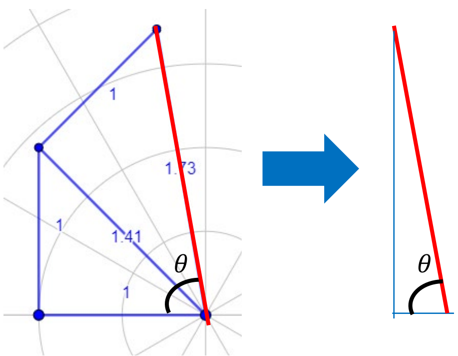
From this point, I realized how TWT seemingly approximates the Archimedean spiral, since the change in radius of spirals, too, approaches to 0 and has virtually no distinct angle as the spiral endlessly progresses (FerréolRobert, 2016).

Erich Teuffel's proposal:

In 1958, Erich Teuffel proposed that while triangles may overlap, no two hypotenuses will ever coincide, regardless of the progression of the spiral (Nelli, n.d.). This is what I came up with to prove his proposal, which is different to his proof (Davis, Iserles, & Gautschi, 1993) as I could not fully comprehend his proof with my limited knowledge of mathematics.

Before getting into the proof, we first need to know how to calculate the gradient of the hypotenuse of a triangle, as having same gradients will indicate that the hypotenuses have coincided. Each hypotenuse of a triangle will have its own gradient depending on the centred angle of each triangle. For example, the gradient of the hypotenuse of a triangle can be thought of it as following (see figure 3).

Figure 3: Gradient of the hypotenuse of second triangle (highlighted in red) in terms of tan (θ)



As can be seen in figure 3, the gradient can now be calculated as following:

$$The\ gradient\ for\ figure\ 3 = \frac{\Delta y}{\Delta x} = \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$$

And the angle for this case will be the sum of centred angles of first two triangles (see equation 2).

$$\theta = \sum_{x=1}^{2} \sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)$$

$$\therefore gradient = \tan\left(\sum_{x=1}^{2} \sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)\right)$$

$$\therefore gradient\ of\ hypotenuse\ of\ n^{th}\ triangle = \tan\left(\sum_{x=1}^{n} \sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)\right) \quad \underline{\textbf{(3)}}$$

Now, the Erich Teuffel's proposal can be proved by contradiction using equation 3.

Firstly, assume there are two hypotenuses of two different triangles where they coincide, meaning the gradients of the two hypotenuses are identical.

$$let\ gradient\ of\ n^{th}\ triangle = gradient\ of\ m^{th}\ triangle, where\ n \neq m$$

$$\therefore \tan\left(\sum_{x=1}^{n} \sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)\right) = \tan\left(\sum_{x=1}^{m} \sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)\right) \quad \underline{\textbf{(4)}}$$

Since $\tan(\theta) = \tan(\theta + k\pi)\ , where\ k \in \mathbb{Z}, to\ satisfy\ the\ assumption\ that\ n \neq m,$

$$\sum_{x=1}^{n} \sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right) = \sum_{x=1}^{m} \sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right) + k\pi$$

$$\sum_{x=1}^{n} \sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right) - \sum_{x=1}^{m} \sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right) = k\pi$$

For the sake of simplicity, assume not only that n is not equal to m, but is also bigger than m. Hence, the equation above can be further simplified into:

$$\left(\cancel{\sin^{-1}\left(\frac{1}{\sqrt{1+1}}\right)} + \cancel{\sin^{-1}\left(\frac{1}{\sqrt{2+1}}\right)} + \cdots + \cancel{\sin^{-1}\left(\frac{1}{\sqrt{m+1}}\right)} + \sin^{-1}\left(\frac{1}{\sqrt{(m+1)+1}}\right) + \cdots + \sin^{-1}\left(\frac{1}{\sqrt{n+1}}\right)\right)$$

$$-\left(\cancel{\sin^{-1}\left(\frac{1}{\sqrt{1+1}}\right)} + \cancel{\sin^{-1}\left(\frac{1}{\sqrt{2+1}}\right)} + \cdots + \cancel{\sin^{-1}\left(\frac{1}{\sqrt{m+1}}\right)}\right) = k\pi$$

$$\sum_{x=m+1}^{n} \sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right) = k\pi \quad \underline{\textbf{(5)}}$$

Since, m used to be the upper boudnary of the sigma notation with the lower boundary of 1 (see equation 4), m must be at least 1. This means that n must be at least 2, as $n \geq m+1$ (see equation 5). Hence, this proof essentially comes down to whether the sum of $\sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)$, where the range of x is between any integers that are both at least 2, can be multiple of π. From this point, I was not sure whether this equation was possible or impossible was to be satisfied, but according to theorem of Sylvester-Schur, there is no n and m that satisfy equation 5 (Davis, Iserles, & Gautschi, 1993). In other words, it is impossible to satisfy equation 4 with two different integers. Hence, it is also impossible for two different hypotenuses to have same gradient, falsifying my assumption and proving Erich Teuffel's proposal.

Winding distance:

To corroborate whether TWT actually shares any common property with the Archimedean spiral, I decided to work out the winding distances (see Figure 4), as any ray from the origin intersects successive revolution of the Archimedean spiral in the points with a constant separation of $2\pi b$ (Ferréol, 2016).

Figure 4: Diagram explaining the definition of the winding distance

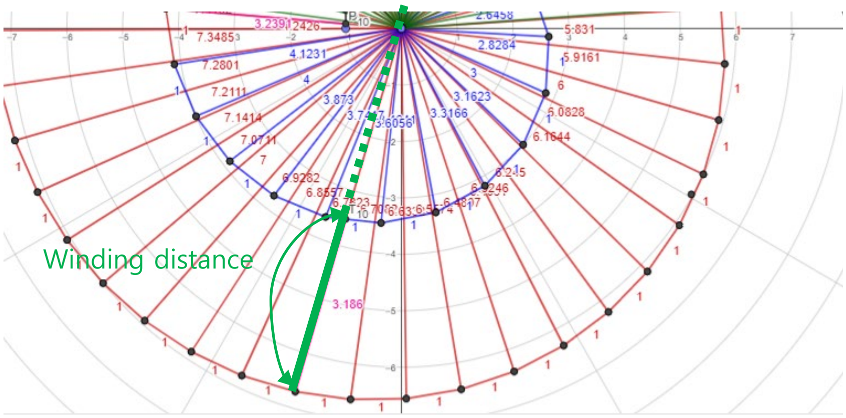
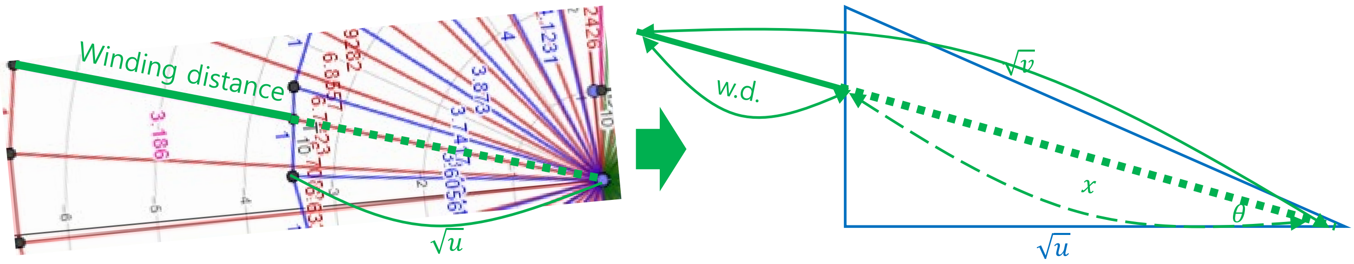


Figure 5: Simplified diagrams for a position of the winding distance in relative to the u^th triangle



$$the\ winding\ distance\ of\ v^{th}\ triangle\ (w.d) = \sqrt{v} - x,$$

Clearly, the winding distance of (v-1)^th triangle (w.d) is subtraction of the dotted line from the radius of the triangle (see Figure 5). It should also be noted that the dotted line is the hypotenuse of the (u-1)^th triangle, which will be expressed as x in this equation (see Figure 5). They are (v-1)^th and (u-1)^th triangles for following reason:

Equation 1: $hypotenuse\ of\ n^{th}\ triangle = \sqrt{1+n}$

Hence, hypotenuses of (v-1)^th and (u-1)^th triangles are $\sqrt{v}$ and $\sqrt{u}$, respectively.

Coming back to the calculation of winding distance, x can also be expressed as following:

$$\cos\theta = \frac{\sqrt{u}}{x}$$

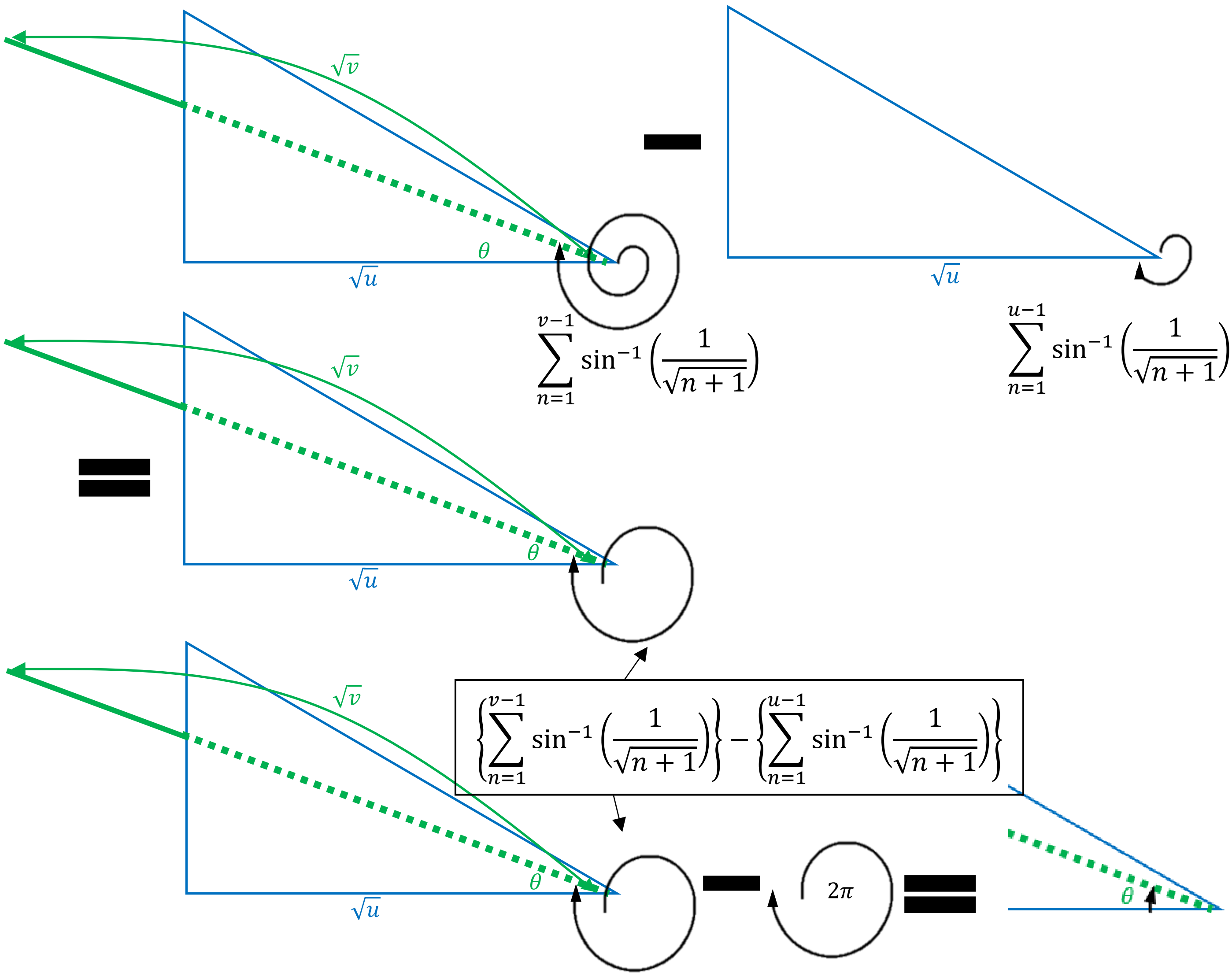
$$\rightarrow x = \frac{\sqrt{u}}{\cos\theta}$$

Substitute $x = \frac{\sqrt{u}}{\cos\theta}$ into the equation of winding distance of (v-1)^th triangle,

$$\therefore\ w.d(v) = \sqrt{v} - \frac{\sqrt{u}}{\cos\theta}$$

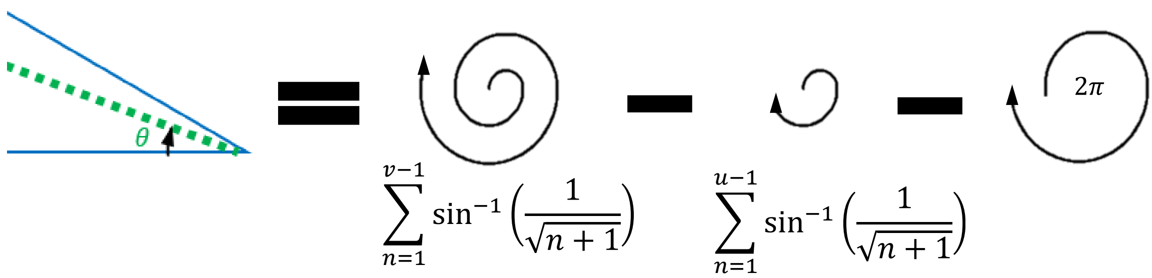
Getting to the equation above did not take me so long; However, thinking a way to calculate the angle (θ) certainly did. I eventually realized that the angle is essentially the subtraction of sum of smaller angles (so up to (u-1)^th triangle) from the larger angles (so up to (v-1)^th triangle) (see Figure 6). The difference between two angles will obviously have extra revolution, since $\sqrt{u}$ is the intersection of $\sqrt{v}$'s successive winding. Thus, the extra revolution (2π) also needs to be subtracted to calculate the real leftover (θ) (see Figure 6).

Figure 6: Equations with winding distance diagrams from Figure 5



Note that the spiral for each diagram represents how much angle TWT have progressed, so figure 6 can even be simplified further as can be seen in figure 7.

Figure 7: Explanation of the angle needed for winding distance in terms of spiral



$$\therefore \theta(\text{the angle between } \sqrt{u} \text{ and } \sqrt{v}) = \left\{\sum_{n=1}^{v-1} \sin^{-1}\left(\frac{1}{\sqrt{n+1}}\right)\right\} - \left\{\sum_{n=1}^{u-1} \sin^{-1}\left(\frac{1}{\sqrt{n+1}}\right)\right\} - 2\pi$$

Coming back to the equation for the winding distance,

$$w.d(v) = \sqrt{v} - \frac{\sqrt{u}}{\cos\theta}$$

$$\therefore w.d\ (v) = \sqrt{v} - \frac{\sqrt{u}}{\cos\left[\left\{\sum_{n=1}^{v-1} \sin^{-1}\left(\frac{1}{\sqrt{n+1}}\right)\right\} - \left\{\sum_{n=1}^{u-1} \sin^{-1}\left(\frac{1}{\sqrt{n+1}}\right)\right\} - 2\pi\right]}$$

I realized that this equation, especially the angle of cosine, could further be simplified.

$$\theta = \left\{\sum_{n=1}^{v-1} \sin^{-1}\left(\frac{1}{\sqrt{n+1}}\right)\right\} - \left\{\sum_{n=1}^{u-1} \sin^{-1}\left(\frac{1}{\sqrt{n+1}}\right)\right\} - 2\pi$$

Using the fact $v > u$, as v[th] triangle is positioned at the next revolution of u[th] triangle,

$$\theta = \left\{\cancel{\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)} + \cancel{\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)} + \cdots + \cancel{\sin^{-1}\left(\frac{1}{\sqrt{u}}\right)} + \sin^{-1}\left(\frac{1}{\sqrt{u+1}}\right) + \cdots + \sin^{-1}\left(\frac{1}{\sqrt{v}}\right)\right\}$$

$$- \left\{\cancel{\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)} + \cancel{\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)} + \cdots + \cancel{\sin^{-1}\left(\frac{1}{\sqrt{u}}\right)}\right\} - 2\pi$$

$$= \sin^{-1}\left(\frac{1}{\sqrt{u+1}}\right) + \cdots + \sin^{-1}\left(\frac{1}{\sqrt{v}}\right) - 2\pi = \sum_{n=u+1}^{v} \sin^{-1}\left(\frac{1}{\sqrt{n}}\right) - 2\pi$$

$$\therefore w.d\ (v) = \sqrt{v} - \frac{\sqrt{u}}{\cos\left[\sum_{n=u+1}^{v} \sin^{-1}\left(\frac{1}{\sqrt{n}}\right) - 2\pi\right]}$$

$$\therefore w.d\ (v) = \sqrt{v} - \frac{\sqrt{u}}{\cos\left[\sum_{n=u+1}^{v} \sin^{-1}\left(\frac{1}{\sqrt{n}}\right)\right]}, \text{as } \cos(\theta) = \cos(\theta - 2\pi) \qquad \textbf{(6)}$$

To corroborate whether equation 4 is, in fact, the right equation to calculate the winding distance, random vertex of a triangle was chosen, which was 46[th]. Hence v for the sample calculation is 46, which automaticallly uses 13 as its u, according to the diagram I constructed.

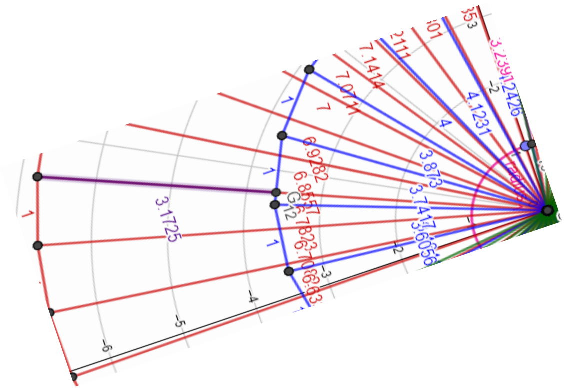
$$substitute\ v = 46, u = 13\ into\ equation\ 4,$$

$$w.d\ (46) = \sqrt{46} - \frac{\sqrt{13}}{\cos\left[\sum_{n=13+1}^{46} \sin^{-1}\left(\frac{1}{\sqrt{n}}\right)\right]}$$

$$\approx 3.1725\ (to\ 5\ sig\ figs)$$

This was, in fact, found to be true, when the winding distance of 46th triangle was computed in the Geogebra diagram (see Fig. 14).

<u>Fig. 14: The winding distance of 46th triangle (highlighted in purple) matches to be 3.1725</u>



10 vertexes were randomly chosen and followed the same procedure as above. Their winding distances were all identical with the values that were deduced from the equation. Of course, there is a chance that all the example vertexes that I chose may have had luckily identical winding distance as what I calculated, but the fact that every single one of them were identical to my calculation in 5 significant figures somewhat mitigate this chance. Note that there is no winding distance for first revolution as there are no intersections of successive revolution.

Also, the downside of this equation is that both a value of 'v' and 'u' must be known. There was an attempt to find a correlation between 'v' and 'u', as I have created a table that shows how many 'v's correspond to each 'u' (see Table 2) by manually matching 'v's and 'u's from the final product of figure 2. The starting 'v' and 'u' was 18 and 1, as they were the values for the first winding distance.

Table 2: Number of v that fit into each u for triangles from 2nd revolution (from n = 18 to 54) and 3rd revolution (from n = 55 to 110)

v	u	v	u	v	u	v	u	v	u	v	u
18~21	1	40~41	10	57~58	19	72	28	86	37	99~100	46
22~24	2	42~43	11	59	20	73~74	29	87~88	38	101	47
25~26	3	44~45	12	60~61	21	75	30	89	39	102	48
27~29	4	46~47	13	62~63	22	76~77	31	90~91	40	103~104	49
30~31	5	48~49	14	64	23	78~79	32	92	41	105	50
32~33	6	50~51	15	65~66	24	80	33	93~94	42	106~107	51
34~35	7	52	16	67~68	25	81~82	34	95	43	108	52
36~37	8	53~54	17	69	26	83	35	96~97	44	109~110	53
38~39	9	55~56	18	70~71	27	84~85	36	98	45		

As the sequence shows, the correlation between ‘v’ and ‘u’ is not completely random. However, it also does not seem to have a mathematical pattern from the perspective of inductive reasoning, unless the sequence above was the repeating pattern of the entire sequence. Such limited knowledge of determining ‘u’ based on ‘v’ meant that the winding distance of each point was limited to the progress of TWT (see the final product of figure 2). It took me awhile to manually construct TWT on Geogebra and for the sake of time saving, I constructed TWT up to 3rd revolution. Because my TWT stopped at the end of 3rd revolution, the following table only contains the winding distance of vertexes in the 2nd and 3rd revolution to verify whether the average winding distance of each revolution approaches to π. Again, to be able to calculate winding distance without relying on the diagram, the equation for winding distance must be expressed as 1 variable, which means algebraic relationship between u and v must be known to express u in terms of v.

Table 3: Winding distances of the bases of triangles from 2nd revolution (from n = 18 to 54) and 3rd revolution (from n = 55 to 110)

n^{th} base	w.d	n^{th} base	w.d	n^{th} base	w.d	n^{th} base	w.d	n^{th} base	w.d
18	3.2391	37	3.2001	55	3.1688	74	3.1643	93	3.1591
19	3.3072	38	3.1640	56	3.1700	75	3.1689	94	3.1528
20	3.3059	39	3.1967	57	3.1760	76	3.1499	95	3.1643
21	3.2023	40	3.1619	58	3.1548	77	3.1667	96	3.1599
22	3.2503	41	3.1934	59	3.1770	78	3.1660	97	3.1502
23	3.2604	42	3.1634	60	3.1615	79	3.1498	98	3.1632
24	3.1789	43	3.1901	61	3.1705	80	3.1674	99	3.1609
25	3.2357	44	3.1673	62	3.1722	81	3.1632	100	3.1465
26	3.2273	45	3.1860	63	3.1553	82	3.1532	101	3.1615
27	3.1927	46	3.1725	64	3.1743	83	3.1672	102	3.1620
28	3.2279	47	3.1803	65	3.1611	84	3.1610	103	3.1487
29	3.1790	48	3.1775	66	3.1668	85	3.1548	104	3.1589
30	3.2138	49	3.1719	67	3.1714	86	3.1666	105	3.1626
31	3.1990	50	3.1810	68	3.1484	87	3.1595	106	3.1530
32	3.1966	51	3.1598	69	3.1710	88	3.1552	107	3.1552
33	3.2041	52	3.1818	70	3.1650	89	3.1659	108	3.1624
34	3.1814	53	3.1566	71	3.1588	90	3.1589	109	3.1569
35	3.2031	54	3.1786	72	3.1710	91	3.1545	110	3.1500
36	3.1704	$\bar{x}$: 3.1989	% to π: 98.210	73	3.1574	92	3.1652	$\bar{x}$: 3.1614	% to π: 99.374

Archimedean spiral in respect to TWT

Now that TWT is known to approximate Archimedean spiral, I wanted to express the wheel in a form of Archimedean spiral equation: $r = a + b\theta,$

$$where\ r = the\ radius\ from\ the\ origin, a = the\ starting\ point\ of\ the\ spiral\ and$$

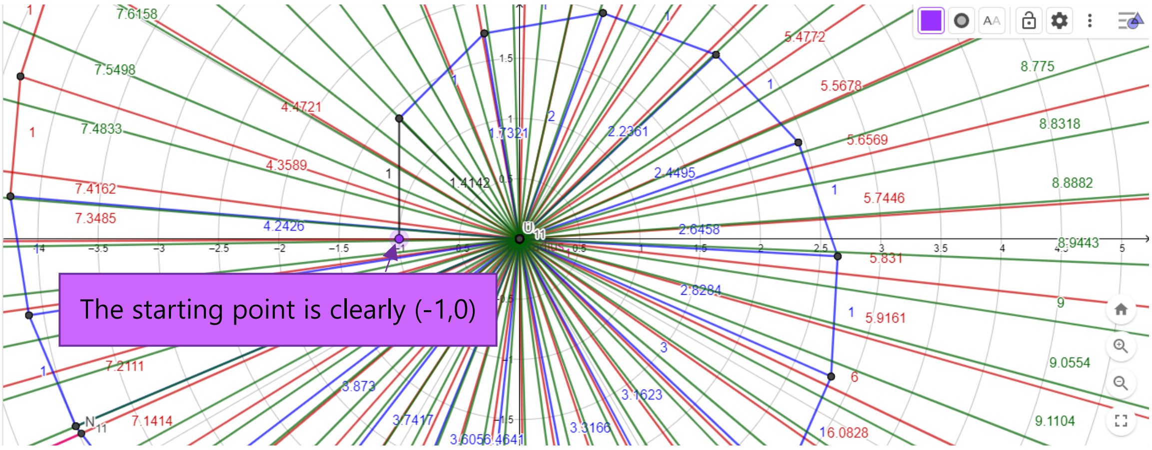
$$b = the\ winding\ distance\ factor\ (where\ w.d = 2\pi b)$$

As the equation implies, Archimedean spiral is the locus corresponding to the locations over time of a point moving away from a fixed point with constant speed along a line that rotates with constant angular velocity.

The parameter 'a' is the starting point of the spiral, as this is the point when the angle is 0: $r = a + b(0) = a$

So, the value of a for TWT that I constructed is -1, since the TWT starts at (-1,0).

Figure 8: Magnified TWT indicating the starting point



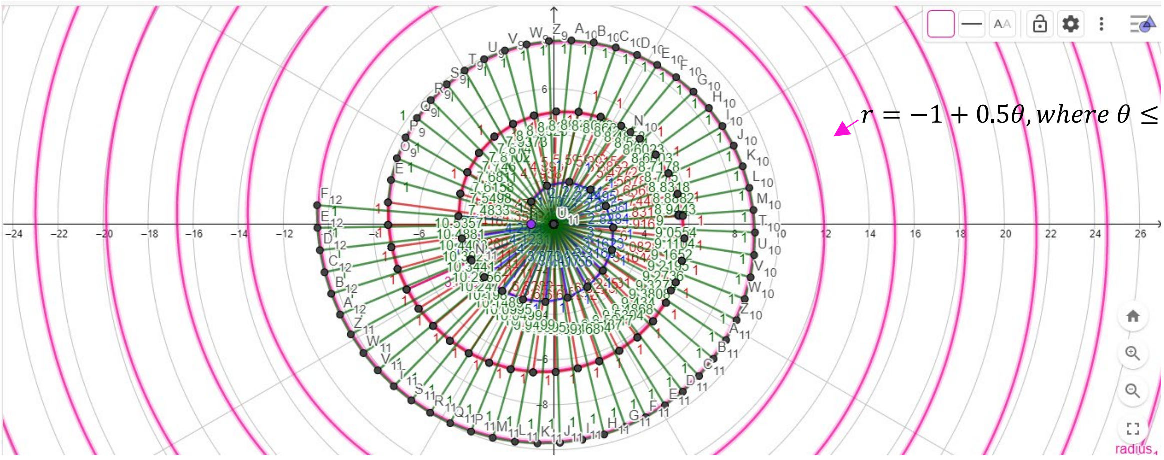
Since we know that the average winding distance of TWT is approaching π and the parameter of the equation 'b' represents the winding distance where the w.d. is $2\pi b$, b is 0.5 $(2\pi \times 0.5 = \pi)$.

$$Substituting\ a = -1, b = 0.5\ into\ r = a + b\theta,$$

$$\therefore TWT\ represents\ the\ archimedean\ spiral\ with\ the\ equation\ of\ r = -1 + 0.5\theta \qquad \underline{(7)}$$

The Archimedean spiral was also drawn on top of TWT (see Figure 9). It should be noted that the range of angle is less than 0, as TWT that I constructed rotates clockwise, which is opposite to the conventional direction of an angle around the unit circle.

Figure 9: TWT overlaid with an Archimedean spiral (pink spiral) to illustrate how TWT approximates the spiral



As the figure 9 shows, TWT, indeed, seems to be reasonably accurate approximation of the Archimedean spiral, as all the vertexes of TWT is approximately aligned with the Archimedean spiral. To verify this without taking too much time, I chose random 10 random hypotenuses from TWT and compared each of them with the corresponding radii of the Archimedean spiral. Following calculations took place for each hypotenuse and its corresponding radii.

The hypotenuse of 3rd triangle (see equation 1): $\sqrt{3+1} = 2$

The total angle up to this hypotenuse (see equation 2):

$$\theta = \sum_{x=1}^{3} \sin^{-1}\left(\frac{1}{\sqrt{3+1}}\right) \approx 1.9245$$

Substitute the value of θ. Note that the negative value of θ was substituted the progression of TWT and the spiral is in clockwise.

$$Equation\ 7: r = -1 + 0.5\theta$$

$$r = -1 + 0.5 \times (-1.9245) = -1.9622$$

$$|r| = 1.96, as\ radius\ must\ be\ positive$$

We can consider hypotenuse of TWT as an experimental value and the radius of the spiral as an accepted value, as we are trying to show that TWT is approximating the spiral. Hence the percentage error the 3rd hypotenuse is as following.

$$Percentage\ error\ (\%) = \left|\frac{Experimental\ value - Accepted\ value}{Accepted\ value}\right|$$

$$= \left|\frac{2 - 1.9622}{1.9622}\right|$$

$$= 1.9244\ \%$$

Therefore, the following table was created to indicate TWT's approximation to the Archimedean spiral with very low percentage errors of the hypotenuses that become almost negligible for big value of n (see table 4).

Table 4: Percentage errors of n^{th} hypotenuses with their corresponding radii

n^{th} hypotenuse	Percentage error with the corresponding radius (%)	n^{th} hypotenuse	Percentage error with the corresponding radius (%)
3	1.9244	200	0.26732
17	1.4398	400	0.24935
39	0.22090	600	0.22541
68	0.10263	800	0.20646
92	0.19081	1000	0.19153

Note that this percentage error is independent of the progression of TWT I have constructed, as only 1 variable, n in n^{th} hypotenuse, is required.

Conclusion

What started off with a pure interest to explore the visualization of surds ended up proving how TWT approximates the Archimedean spiral, by investigating many components of the wheel. I am particularly satisfied by the fact that I invented my own equation to calculate the winding distance, which I decided to make since there was simply no information about the calculation of the winding distance in online. In this respect, it was very interesting to approach the properties of TWT and its relatedness to the Archimedean spiral with the use of moderately complex mathematics, rather than drawing out deductive conclusion through the use of complex mathematics that is beyond the high school level, such as utilising a gamma function to interpolate the discrete points of TWT by a continuous curve or applying the Riemann zeta function to calculate K, which is used to find the bound of the angles' growth (Kociemba, 2018). Furthermore, the diagram of TWT helped me significantly to not only check if the calculated values were right, but also to formulate equation, as the visual representation made the comprehension of the properties much more accessible. Nevertheless, Excel was utilised effectively to calculate all the values with only few functions. However, I could not figure out a way to carry out calculation involving trigonometry inside sigma notation in Excel, which then took me a long time to manually calculate each value with a graphics calculator. For future investigation, I would look more into the correlation between 'u' and 'v' of the winding distance by considering the total angle of each revolution. To conclude, I learnt a lot about the fundamental concept of trigonometry through the exploration of TWT, although the level of math may not have been at the most challenging level. Perhaps, the fact that I now understand the essence of less complex mathematics will allow me to understand more complex concepts easier in future.

Bibliography

Davis, P. J., Iserles, A., & Gautschi, W. (1993). Spirals: From Theodorus to Chaos. In P. J. Davis, A. Iserles, & W. Gautschi, *Spirals: From Theodorus to Chaos* (pp. 10,160-167). Wellesley: A K Peters/CRC Press.

Ferréol, R. (2016, n.d. n.d.). *ARCHIMEDEAN SPIRAL*. Retrieved from Math Curve: https://mathcurve.com/courbes2d.gb/archimede/archimede.shtml

Fractal Kitty. (2021). *Week 2: Spiral of Theodorus*. Retrieved from Fractal Kitty: https://fractalkitty.com/2019/06/12/week-2-spiral-of-theodorus/

Kociemba, H. (2018). *The Spiral of Theodorus*. Retrieved from Kociemba: http://kociemba.org/themen/spirale/theodorus.html

Nelli, E. (n.d.). *Spiral of Theodorus*. Retrieved from The University of Georgia: http://jwilson.coe.uga.edu/EMAT6680Fa2012/Nelli/spiraloftheodorus/spiraloftheodorus.html